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field pattern side-lobe level. In a radar system, the high-resolution and reduction in ambiguity is achieved at the expense of a reduction in power in the main beam when compared with a densely packed array of the same aperture length.

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On the Closed Form of the Array Factor for Linear Arrays*

Recently Cheng and Ma[1] have shown that known relations in Z-transform theory developed for sampled-data systems can be used to express the array polynomial for a linear array in a closed form.

The Z-transform of a real function $f(t)$ is defined by

$$Z\{f(t)\} = \sum_{\nu=0}^{\infty} f(\nu T) z^{-\nu} = F(z). \quad (1)$$

The array factor for a n-element array is

$$\sum_{\nu=0}^{n-1} \sigma(\nu d) z^{-\nu} = G(z) \quad (2)$$

where $\sigma(x)$ is the envelope function for the current distribution, and d is the distance between adjacent elements.

When a nonlinear phase envelope function is not included in $\sigma(x)$, $\sigma(x)$ becomes real, and the analogy is established by replacing t by x, T by d and $f(t)$ by $\sigma(x)$. However, the summation is extended to infinity in (1), but not in (2).

Cheng and Ma[1] have overcome this difficulty by assuming

$$\sigma(x) = \sigma(x + (n-1)d) \quad (3a)$$

or

$$\sigma(x) = -\sigma(x + (n-1)d) \quad (3b)$$

yielding, respectively,

$$G(z) = (1 - z^{-(n-1)}) \cdot Z\{\sigma(x)\} + \sigma(0) z^{-(n-1)} \quad (4a)$$

and

$$G(z) = (1 + z^{-(n-1)}) \cdot Z\{\sigma(x)\} - \sigma(0) z^{-(n-1)}. \quad (4b)$$

This assumption, however, restricts the envelope functions to periodic ones.

The following alternative procedure removes this limitation. We introduce a unit gate function $\gamma_n(x)$ defined by

$$\gamma_n(x) = \begin{cases} 0 & x < 0 \text{ and } x > (n-1)d \\ 1 & 0 \leq x \leq (n-1)d \end{cases} \quad (5)$$

thereby obtaining the more general formula

$$G(z) = Z\{\sigma(x)\gamma_n(x)\} \quad (6)$$

in which the array factor is expressed as a Z-transform.

The usefulness of formula (6) can be illustrated by an example. Consider the envelope function

$$\sigma(x) = (\omega x)^p e^{-ax} \quad (7)$$

where p is a nonnegative integer, ω a real number, and a is any real or complex number (yielding envelope functions including trigonometric sine and cosine).

For this envelope function formula (6) yields the array factor

$$G(z) = Z\{(\omega x)^p e^{-ax} \gamma_n(x)\} = (-\omega)^p \frac{\partial^p}{\partial a^p} \frac{1 - e^{-and} z^{-n}}{1 - e^{-ad} z^{-1}} \quad (8)$$

which is closed in n, but not in p. By means of (8) many useful current distributions can be investigated.

Cheng and Ma[1] have shown that when the current distribution does not involve a nonlinear phase envelope function the numerator and the denominator in the expression for $|G(z)|^2$ always contain the terms z^m and z^{-m} in pairs $z^m + z^{-m}$.

Using

$$z = e^{-j\psi} \quad (9)$$

they introduce

$$y = z + z^{-1} = 2 \cos \psi \quad (10)$$

and make use of the fact that every pair $z^m + z^{-m}$ can be evaluated as a polynomial in y.

In the more general case (not considered by Cheng and Ma),[1] where the current envelope function $\sigma(x)$ includes an amplitude envelope function $r(x)$ and a nonlinear phase envelope function $\phi(x)$,

$$\sigma(x) = r(x) e^{j\phi(x)} = a(x) + jb(x). \quad (11)$$

Eq. (2) can be written

$$G(z) = \sum_{\nu=0}^{n-1} a(\nu d) z^{-\nu} + j \sum_{\nu=0}^{n-1} b(\nu d) z^{-\nu} = Z\{a(x)\gamma_n(x)\} + jZ\{b(x)\gamma_n(x)\}. \quad (12)$$

We have found it useful to apply the following relations:

$$z^m + z^{-m} = P_m(y) = 2T_m\left(\frac{y}{2}\right), \quad m = 0, 1, \cdots \quad (13)$$

$$j(z^m - z^{-m}) = Q_m(y) = \pm 2U_m\left(\frac{y}{2}\right), \quad m = 1, 2, \cdots \quad (14)$$

Plus sign for $\psi \geq 0$;
Minus sign for $\psi \leq 0$;
where T_m and U_m are the Chebyshev functions of the first and second kind.[2]

It turns out that in this case the denominator in the expression for $|G(z)|^2$ still may be written as a linear combination of Chebyshev functions of the first kind, while the numerator now also contains linear combinations of Chebyshev functions of the second kind.

It can be shown that if one of the following two conditions,

$$r(x) = r((n-1)d - x) \quad (15a)$$

$$r(x) = -r((n-1)d - x), \quad (15b)$$

and the condition

$$\phi(x) = \phi((n-1)d - x) \quad (16)$$

are satisfied, $|G(z)|^2$ contains no Chebyshev functions of the second kind.

We have investigated several distributions with complex envelope functions as in (11), *e.g.*,

$$\sigma(x) = (1 + \sin^2 \omega x) \cdot e^{j2 \tan^{-1} \sin \omega x} = \cos^2 \omega x + j \cdot 2 \sin \omega x \quad (17)$$

(no Q_m-functions are involved in $|G(z)|^2$ in this case, if $\omega(n-1)d = 1$, because (15a) and (16) are both satisfied) and

$$\sigma(x) = \cosh(\omega x + \beta) e^{j gd(\omega x + \beta)} = 1 + j \sinh(\omega x + \beta), \quad (18)$$

where gd is the Gudermannian angle.

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[1] D. K. Cheng and M. T. Ma, "A new mathematical approach for linear array analysis," IRE Trans. on Antennas and Propagation, vol. AP-8 pp. 255–259; May, 1960.

[2] T. M. Korn and G. A. Korn, "Mathematical Handbook for Scientists and Engineers," McGraw-Hill Book Co., Inc., New York, N. Y., p. 723; 1961.

Designing for Desired Aperture Illuminations in Cassegrain Antennas*

When designing microwave reflector antennas, one of the major goals is to obtain the desired illumination in the antenna aperture. In Cassegrain antennas, the aperture field distribution depends on shape and size of both horn and subreflector, and this apparently makes the design of a desired illumination more involved than in a conventional antenna system using a simple feed horn at the focus. However, a closer look at this problem reveals a possibility to determine the aperture field (and thus also the radiation pattern) without detailed information about horn and subreflector. The only information needed is the type of horn that is to be used, as will be shown below.

The basic geometry of the Cassegrain system is shown in Figs. 1 and 2, as well as the notations used. For simplicity, we assume the horn pattern $g(\phi)$ to be symmetrical, in which case the illumination

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